Two-Pass Method for Handling Difficult Equality Constraints in Optimization

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In the optimization of engineering systems, constraints are frequently encountered in the form of equalities as well as inequalities. In many instances, the equality constraints can be readily handled simply by solving for a particular variable. The question of what to do if such simple rearrangement is not possible has not received adequate attention in the literature. An equality which does not allow one variable to be solved readily in terms of the other variables is termed a *difficult equality*. The purpose of this note is to present an effective procedure to handle difficult equality constraints in nonlinear programming problems.

A SIMPLE GEOMETRIC PROBLEM

Let us consider a simple geometrical problem which is easy to visualize. Let us take the intersection of the ellipsoid

$$4(x_1 - 0.5)^2 + 2(x_2 - 0.2)^2 + x_3^2 + 0.1\, x_1x_2 + 0.2\, x_2x_3 = 16 \quad (1)$$

with the hyperboloid

$$2x_1^2 + x_2^2 - 2x_3^2 = 2 \quad (2)$$

and consider the problem of determining the maximum distance from the origin to the intersection of the ellipsoid with the hyperboloid. We may consider the square of the distance and thus formulate the problem as the maximization of

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \quad (3)$$

subject to the constraints of Equations (1) and (2). To keep the problem as simple as possible to visualize, no inequality constraints are added.

The constraints given by Equations (1) and (2) represent difficult equalities since in either equation a variable cannot be solved uniquely in terms of the other variables. The feasible region can be pictured as two isolated ellipses in space. On each ellipse we have two maxima and two minima. All together, there are thus four maxima and four minima, and it is required to find the global maximum.

With the use of Lagrange multipliers, the problem can be readily handled by the procedure of Luus and Jaakola (1973b) to yield the results of Table 1 where also the frequency of getting the 8 stationary conditions from 100 starting points taken at random in the range $-5 \leq x_i \leq 5$ is given. The global maximum of $f = 11.67664$ is thus readily established. However, the aim of this note is to show that the global maximum can be established even more easily, without the introduction of Lagrange multipliers, by the direct search method of Luus and Jaakola (1973a), if the problem is solved in two passes. Furthermore, with the latter method, inequality constraints may be added without introducing any difficulties.

TABLE 1. STATIONARY CONDITIONS FOR THE AUGMENTED PERFORMANCE INDEX: $I = f + \lambda_1\varphi_1 + \lambda_2\varphi_2$

f	x_1	x_2	x_3	No. of times of obtaining local extremum
11.68*	0.988	2.674	−1.884	20
10.47	1.040	2.492	1.785	12
9.57	1.376	−2.126	1.776	12
8.93	2.224	−0.157	1.989	8
8.78	1.568	−1.801	−1.755	7
8.72	2.043	−0.956	−1.906	9
3.06	−1.419	−0.161	1.013	13
3.04	−1.405	−0.245	−1.002	19

* Global maximum, $f = 11.67664$

REPLACEMENT OF DIFFICULT EQUALITIES BY INEQUALITIES

Suppose we write the constraint given by Equation (1) by

$$\varphi_1(x_1, x_2, x_3) = 0 \quad (4)$$

By introducing two auxiliary variables α_1 and β_1, Equation (4) may be replaced by two inequalities

$$\varphi_1(x_1, x_2, x_3) + \alpha_1 \geq 0 \quad (5)$$

$$\varphi_1(x_1, x_2, x_3) - \beta_1 \leq 0 \quad (6)$$

where the auxiliary variables are constrained by

$$\alpha_1 \geq 0 \quad (7)$$

$$\beta_1 \geq 0 \quad (8)$$

Similarly, Equation (2) may be written as

$$\varphi_2(x_1, x_2, x_3) + \alpha_2 \geq 0 \quad (9)$$

$$\varphi_2(x_1, x_2, x_3) - \beta_2 \leq 0 \quad (10)$$

with

$$\alpha_2 \geq 0 \quad (11)$$

$$\beta_2 \geq 0 \quad (12)$$

Let us suppose that the auxiliary variables are all chosen to be of finite magnitude; let us, for example, take $\alpha_k = 1$, $\beta_k = 1$, $k = 1,2$. The feasible region now has been enlarged from the two ellipses to two elliptical donut-shaped regions. The feasible region is thus enlarged considerably and the optimization can be readily carried out by the method suggested by Luus and Jaakola (1973a).

The initial center point was taken to be the origin and the size of the initial region for random numbers was taken as 10 for each variable. For each iteration, 100 random numbers were chosen inside the specified regions and 1000 iterations were allowed. After each iteration, the size of each region was reduced by 1%. Also, for the first 100 iterations, the center point of the region was kept at the origin to ensure that a leap to the wrong donut would not occur.

By using this procedure, a maximum value for f of

12.56 was obtained in a computation time of 6 sec. on an IBM 370/165 digital computer. The values for x_1, x_2, and x_3 are 1.02, 2.71, and −2.05, respectively. But of greater importance is to observe that at this optimum point, Equations (6) and (9) are tight; that is, the equalities hold.

REMOVAL OF APPROXIMATIONS BY SECOND PASS

Since Equations (6) and (9) are tight, to remove any errors due to the approximations, we must rerun the problem while putting $\beta_1 = 0$ and $\alpha_2 = 0$. The problem was thus rerun by the same optimization procedure and starting at the origin as before. After 1000 iterations requiring 5 sec. of computer time we obtained $f = 11.67656$ with $x_1 = 0.981$, $x_2 = 2.677$, and $x_3 = -1.883$. The computation time here is less since the size of the feasible region is halved.

The left-hand sides of Equations (6) and (9) were −0.00001 and 0.00008, respectively, and of Equations (5) and (10), 0.99999 and −0.99992, respectively.

It should be noted that the maximum obtained for f with this procedure is very slightly less than the true global maximum of 11.67664 and that the original equations of the ellipsoid and the hyperboloid are satisfied within 10^{-4}.

GENERALIZATION OF THE TWO-PASS PROCEDURE

Consider the nonlinear programming problem as follows:

Maximize

$$P = f(x_1, x_2, \ldots, x_n) \tag{13}$$

subject to the inequality constraints

$$g_i(x_1, x_2, \ldots, x_n) \leq 0, \; i = 1, 2, \ldots, m \tag{14}$$

$$h_j(x_1, x_2, \ldots, x_n) \geq 0, \; j = 1, 2, \ldots, r \tag{15}$$

and the difficult equality constraints

$$q_k(x_1, x_2, \ldots, x_n) = 0, \; k = 1, 2, \ldots, s \tag{16}$$

where $s < n$. It is assumed that all simple equality constraints have been removed by substitution, thereby reducing the dimension of the problem to n.

First Pass

Replace each equality in Equation (16) by two inequalities

$$\left.\begin{array}{l} q_k(x_1, x_2, \ldots, x_n) + \alpha_k \geq 0 \\ q_k(x_1, x_2, \ldots, x_n) - \beta_k \leq 0 \end{array}\right\} \tag{17}$$

with

$$\left.\begin{array}{l} \alpha_k \geq 0 \\ \beta_k \geq 0 \end{array}\right\} \tag{18}$$

Choose positive and finite (that is, reasonably large) values for each α_k and β_k and solve the nonlinear programming problem of maximizing P given by Equation (13), subject to inequality constraints given by Equations (14), (15), and (17).

Second Pass

Identify which of the equations in the set of Equations (17) are tight at the optimum found in Pass 1. In each tight equation put α_k or β_k to zero. Note that this choice puts either α_k or β_k to zero, but not both. Now solve the nonlinear programming problem again to yield the true maximum.

If α_k and β_k are chosen so large that at the optimum in Pass 1 there are no inequalities in the set of Equations (17) which are tight, or if it is very difficult to obtain a feasible solution, it is recommended that an augmented performance index be used in the second pass. The mechanics of the procedure are illustrated by the following example.

Table 2. Values of Parameters for the Multicomponent Blending Problem

i	a_i	b_i	c_i	d_i
1	0.0693	44.094	123.7	31.244
2	0.0577	58.12	31.7	36.12
3	0.05	58.12	45.7	34.784
4	0.20	137.4	14.7	92.7
5	0.26	120.9	84.7	82.7
6	0.55	170.9	27.7	91.6
7	0.06	62.501	49.7	56.708
8	0.10	84.94	7.1	82.7
9	0.12	133.425	2.1	80.8
10	0.18	82.507	17.7	64.517
11	0.10	46.07	0.85	49.4
12	0.09	60.097	0.64	49.1
13	0.0693	44.094		
14	0.0577	58.12		
15	0.05	58.12		
16	0.20	137.4		
17	0.26	120.9		
18	0.55	170.9		
19	0.06	62.501		
20	0.10	84.94		
21	0.12	133.425		
22	0.18	82.507		
23	0.10	46.07		
24	0.09	60.097		

MULTICOMPONENT BLENDING PROBLEM

Consider the application of the proposed two-pass procedure to the problem of minimization of the cost of blending multicomponent mixtures as considered by Himmelblau (1972). The objective function to be minimized is

$$C = \sum_{i=1}^{24} a_i x_i \tag{19}$$

subject to the constraints

$$h_i = \frac{x_{(i+12)}}{b_{(i+12)} \sum_{j=13}^{24} \frac{x_j}{b_j}} - \frac{c_i x_i}{40 b_i \sum_{j=1}^{12} \frac{x_j}{b_j}} = 0 \tag{20}$$

$$i = 1, 2, \ldots, 12$$

$$h_{13} = \sum_{i=1}^{24} x_i - 1 = 0 \tag{21}$$

$$h_{14} = \sum_{i=1}^{12} \frac{x_i}{d_i} + k \sum_{i=13}^{24} \frac{x_i}{b_i} - 1.671 = 0 \tag{22}$$

where $k = (0.7302)(530)(14.7/40)$ and the coefficients a_i, b_i, c_i, and d_i are given in Table 2.

In addition, the inequality constraints to be satisfied are

$$x_i \geq 0 \quad i = 1, 2, \ldots, 24 \tag{23}$$

Although the problem may appear to be complex, it is quite simple since the 12 equations incorporated in

Equation (20) are linear in x_i, $i = 1, 2, \ldots, 12$ once values are given to x_j, $j = 13, 14, \ldots, 24$. Also, Equation (21) is linear. Therefore, we can take the first eleven equations from Equation (20) plus Equation (21) to give us 12 simple equalities which can be solved by Gaussian elimination to yield x_i, $i = 1, 2, \ldots, 12$ once x_j, $j = 13, 14, \ldots, 24$ are assigned some values. However, we are also faced with the two difficult equalities

$$h_{12} = 0 \tag{24}$$

and

$$h_{14} = 0 \tag{25}$$

which can be handled readily with the proposed two-pass procedure.

In Pass 1, in order to enlarge the feasible region, Equation (24) was replaced by the two inequalities

$$h_{12} + 1.0 \geqq 0 \tag{26}$$

$$h_{12} - 1.0 \leqq 0 \tag{27}$$

and Equation (25) was replaced by

$$h_{14} + 1.0 \geqq 0 \tag{28}$$

$$h_{14} - 1.0 \leqq 0 \tag{29}$$

The procedure of Luus and Jaakola (1973a) can now be applied directly. The initial values for x_j, $j = 13, 14, \ldots, 24$ were chosen as 0.04 and the initial region for each of these variables was taken as 0.4. The reduction factor for the regions was taken as 1% and 100 iterations were allowed.

At the end of this first pass, requiring 0.15 min. of computation time, it was found that at the optimum $x_{13} = 0.100$, $x_{14} = 0.382$, $x_{15} = 0.334$, and x_{16} to x_{24} were 0.0. From the 12 simple equalities, we obtained $x_1 = 0.007$, $x_2 = 0.106$, $x_3 = 0.064$, $x_{12} = 0.007$ with the remaining x_i zero. The performance index was 0.0561 with $h_{12} = -0.0006$ and $h_{14} = 0.4093$.

Since the values of h_{12} and h_{14} were quite small, it was decided to use an augmented performance index for the second pass; that is,

$$I = C + .05(\sqrt{|h_{12}|} + \sqrt{|h_{14}|}) \tag{30}$$

where C is given by Equation (19). The optimum from pass 1 was used as a starting point. The initial region for each variable was chosen to be one-half of the initial value of the corresponding variable, except where the variable was zero. Under this condition, the region was chosen to be 0.001. Reduction factor was chosen to be 5% and 100 iterations were allowed.

At the end of pass 2, requiring 0.75 min. of computation time, it was found that $x_{13} = 0.062144$, $x_{14} = 0.256944$, $x_{15} = 0.338707$, $x_{16} = 0.000078$, $x_{17} = 0.000368$, $x_{18} = 0.000000$, $x_{19} = 0.000305$, $x_{20} = 0.000148$, $x_{21} = 0.000865$, $x_{22} = 0.000045$, $x_{23} = 0.000229$, $x_{24} = 0.000165$. From these values the 12 simple equalities yielded $x_1 = 0.010048$, $x_2 = 0.162117$, $x_3 = 0.148238$, $x_4 = 0.000106$, $x_5 = 0.000087$, $x_6 = 0.000000$, $x_7 = 0.000123$, $x_8 = 0.000416$, $x_9 = 0.008237$, $x_{10} = 0.000051$, $x_{11} = 0.005379$, $x_{12} = 0.005199$.

The value of the augmented performance index was 0.0561, and of C was 0.0559; also, it was noted that $h_{12} = -0.000001$ and $h_{14} = 0.000005$ are negligible. This value of the performance index is slightly better than 0.0570 as obtained by Himmelblau (1972, p. 420). The important factor, however, is still the ease with which the results were obtained. Both the programming effort and computation time may be regarded as negligible for this problem.

DISCUSSION

The two-pass procedure as suggested by Luus (1973), and now illustrated here, appears to be a useful means of solving nonlinear programming problems where numerous equality constraints are encountered. The optimization procedure as developed by Luus and Jaakola (1973a) and which was used here is ideally suited for such a two-pass optimization procedure. Furthermore, the nonlinear programming procedure of Bracken and McCormick (1968) is well suited for the suggested procedure since the feasible regions in their problems can be expanded to facilitate obtaining the first feasible solution, and any errors due to the use of inequalities instead of equalities can then be removed.

Although sometimes equality constraints can be rearranged to avoid difficulties altogether as has been shown by Jaakola and Luus (1974), at other times, as illustrated here, some difficult equality constraints remain. The proposed two-pass procedure provides an effective means of handling such problems without the introduction of auxiliary variables, other than the parameters used to convert equalities to inequalities in the first pass.

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NOTATION

a_i = parameter defined in Table 2
b_i = parameter defined in Table 2
c_i = parameter defined in Table 2
C = cost function to be minimized
d_i = parameter defined in Table 2
f = function to be maximized
g_i = general function
h_j = general function
I = augmented performance index
q_k = general function expressing difficult equality constraint
s = number of difficult equality constraints
x_i = variable

Greek Letters

α_k = auxiliary variable, positive in first pass
β_k = auxiliary variable, positive in first pass
λ_k = Lagrange multiplier
φ_k = general function expressing equality constraint

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